Existence of multibreathers in chains of coupled one-dimensional Hamiltonian oscillators

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We prove the existence of monoparametric families of multibreathers in chains of Hamiltonian oscillators of one degree of freedom, with the total energy of the chain as a parameter. At the same time we evaluate the neighborhood of the initial conditions for these solutions, as well as their stability. For the proof we use an idea originally proposed by Poincaré. We apply our results to calculate families of multibreathers in a chain consisting of coupled Morse oscillators.

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I. INTRODUCTION

Since the paper of Takeno *et al.* [1], much work has been done on discrete breathers, i.e., space-localized time-periodic motions, in chains of coupled oscillators; see for example, Refs. [2,3]. The first existence proof for breathers in Hamiltonian time-reversible networks is given in the well-known paper by MacKay and Aubry [4], where they make use of the notion of the anticontinuous limit. The anticontinuous limit of a system, depending on at least one parameter ϵ, is the limit where the system becomes equivalent to a collection of uncoupled oscillators (e.g., $\epsilon=0$). The method consists of the continuation of a trivial periodic and localized motion of the anticontinuous limit with respect to the parameter ϵ. The problem is that, in this case, the implicit function theorem cannot be applied due to the phase degeneracy of the solutions of the system. This problem is overcome in Ref. [4] by restricting the working space on the space of periodic loops with time-reversal symmetry. After this proof, several other works appeared, using and expanding the same idea, which demonstrated the existence of breathers in more complex systems; for example, Refs. [5,6]. Only recently, a proof based on variational methods appeared [7].

In the present work, we prove the existence of multisite or single-site breathers in a chain of coupled one-dimensional Hamiltonian oscillators with on-site potential. In the proof we use a different method of continuation from the anticontinuous limit, where the parameter is the coupling constant ϵ, based on a theorem by Poincaré ([8,9], Sec. 42). The generalization of the same idea for symplectic maps [10] has already been used for the proof of existence of breathers in chains of coupled integrable symplectic oscillators [11]. This limit will also be called ‘‘uncoupled’’ or ‘‘unperturbed case’’ since for $\epsilon\neq 0$ the system is perturbed and becomes nonintegrable. The proof of existence of multibreathers does not require the same conditions as in Ref. [4], but, as it has already been mentioned in Refs. [2] and [12], it makes use of the relative phases between the oscillators, which in the present work are expressed through the resonant angles ϕ_i. The conditions for the continuation from the anticontinuous limit are similar to these in Ref. [12], but they are derived through a different, simpler path, using the idea of Poincaré.

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By our method, the localization of the continued solutions is shown at the same time. Regarding the stability of the continued solutions, we follow a different approach than in Ref. [13]. We directly use the formulas that have been already proved by Poincaré, which determine the linear stability by evaluating the expansion of the characteristic exponents of the periodic solutions up to $O(\sqrt{\epsilon})$ terms. We apply these results to the case where the system is a chain consisting of Morse oscillators with a weak nearest-neighbor coupling. We calculate analytically the ϵ neighborhood of the initial conditions of the breather solutions and determine their stability. This method can be easily extended to oscillators with more than one degree of freedom. Note that this method cannot be applied in dissipative systems, where the existence of multibreathers has already been shown [14], since all the periodic orbits are isolated. But, it can be generalized for Hamiltonian systems with a weak dissipative perturbation.

II. EXISTENCE OF BREATHERS IN A CHAIN CONSISTING OF ONE DEGREE OF FREEDOM OSCILLATORS

We define our oscillator by an autonomous Hamiltonian of one degree of freedom,

$$H_u=\frac{1}{2}p^2+V(x),$$

where $V(x)$ is the potential function. In this case the system is integrable since H_u is always an integral of motion. We assume that $V(x)$ possesses a minimum at $x=0$ with $V''(0)=\omega_p^2$ with $\omega_p\in\mathbb{R}$. We construct our chain by considering a countable set of oscillators with a nearest-neighbor coupling through a coupling constant ϵ. The Hamiltonian then becomes

$$H=H_0+\epsilon H_1=\sum_{i=-\infty}^{\infty}\left(\frac{1}{2}p_i^2+V_i(x_i)\right)+\frac{\epsilon}{2}\sum_{i=-\infty}^{\infty}(x_{i+1}-x_i)^2, \tag{1}$$

where x_i is the position, p_i the momentum, and V_i the potential of the ith oscillator. Note that H_0 is trivially integrable, being separable.

The equations of motion for this Hamiltonian are

$$\dot{x}_k=\frac{\partial H}{\partial p_k}=p_k,$$

$$\dot{p}_k=-\frac{\partial H}{\partial x_k}=-V_k'(x_k)+\epsilon(x_{k+1}-2x_k+x_{k-1}),\quad k\in\mathbb{Z}\tag{2}$$

where $V_k'(x_k)=dV_k(x_k)/dx_k$.

We assume that, for $\epsilon=0$, the three central oscillators move on periodic orbits, satisfying the resonance condition

$$\frac{\omega_{-1}}{k_{-1}}=\frac{\omega_0}{k_0}=\frac{\omega_1}{k_1}=\omega,\tag{3}$$

where ω_i is the frequency of the ith oscillator, while the other oscillators lie on the stable equilibrium. With this assumption, the complete system moves, for $\epsilon=0$, on a nonisolated periodic orbit, lying on a three-dimensional torus, with period $T=k_{-1}T_{-1}=k_0T_0=k_1T_1=2\pi/\omega$, where $T_i=2\pi/\omega_i$. We seek conditions for the continuation of this periodic motion under sufficiently small perturbation. For the proof we will use an idea originally proposed by Poincaré ([8,9], Sec. 42). This method cannot be applied in the case of only one central oscillator moving on a periodic orbit for $\epsilon=0$, for reasons that will be explained later on this work, but we could use any number of central oscillators larger than one. We choose to use three, because it is the smallest number of oscillators that can provide solutions symmetric with respect to the zeroth-site oscillator. In the following, by "central oscillators" we mean all the oscillators that initially (for $\epsilon=0$) move on a periodic orbit.

We know that the solution of a Hamiltonian system, with a Hamiltonian depending analytically on a parameter ϵ, is analytic with respect to this parameter [15]. The three central oscillators move initially on periodic orbits. So, if these motions are continued under small perturbation, they can be expanded in terms of ϵ as

$$x_k=x_k^{(0)}+\epsilon x_k^{(1)}+O(\epsilon^2),$$

$$p_k=p_k^{(0)}+\epsilon x_k^{(1)}+O(\epsilon^2),\quad k\in\{-1,0,1\},\tag{4}$$

where $x_k^{(0)}$ is the unperturbed periodic solution and is considered a known T-periodic function of time. The solution for the other oscillators expands as

$$x_k=0+\epsilon x_k^{(1)}+O(\epsilon^2),$$

$$p_k=0+\epsilon x_k^{(1)}+O(\epsilon^2)\quad(|k|>1),\tag{5}$$

since (for $\epsilon=0$) they lie on the stable equilibrium (0,0).

By inserting the expansions (4) and (5) into the system (2), for the first-order terms, we get

$$\dot{x}_k^{(1)}=p_k^{(1)},\tag{6a}$$

$$\dot{p}_k^{(1)}=-\omega_p^2x_k^{(1)}\quad(|k|>2),\tag{6b}$$

$$\dot{x}_{\pm2}^{(1)}=p_{\pm2}^{(1)},\tag{6c}$$

$$\dot{p}_{\pm2}^{(1)}=-\omega_p^2x_{\pm2}^{(1)}+x_{\pm1}^{(0)},\tag{6d}$$

$$\dot{x}_{\pm1}^{(1)}=p_{\pm1}^{(1)},\tag{6e}$$

$$\dot{p}_{\pm1}^{(1)}=-V_1''(x_{\pm1}^{(0)})x_{\pm1}^{(1)}-2x_{\pm1}^{(0)}+x_0^{(0)},\tag{6f}$$

$$\dot{x}_0^{(1)}=p_0^{(1)},\tag{6g}$$

$$\dot{p}_0^{(1)}=-V_0''(x_0^{(0)})x_0^{(1)}+(x_1^{(0)}-2x_0^{(0)}+x_{-1}^{(0)}).\tag{6h}$$

We define $\eta_k^{(l)}=(x_k^{(l)},p_k^{(l)})^T$. The nontrivial solutions of Eqs. (6a, 6b) are periodic with period $T_p=2\pi/\omega_p$. Since we search for periodic solutions with $T\neq T_p$, we select the trivial solution

$$\eta_k^{(1)}=0,\quad\forall\ |k|>2.\tag{7}$$

So, for these oscillators we have to pursue the analysis further (in higher-order terms). From Eqs. (6c, 6d), for the ±2 oscillators we get

$$\begin{pmatrix}\dot{x}_{\pm2}^{(1)}\\ \dot{p}_{\pm2}^{(1)}\end{pmatrix}=\begin{pmatrix}0&1\\ -\omega_p^2&0\end{pmatrix}\begin{pmatrix}x_{\pm2}^{(1)}\\ p_{\pm2}^{(1)}\end{pmatrix}+\begin{pmatrix}0\\ x_{\pm1}^{(0)}\end{pmatrix},$$

which can be written in the form

$$\dot{\eta}_{\pm2}^{(1)}=A\eta_{\pm2}^{(1)}+\mathbf{f}_{\pm2}(t),\tag{8}$$

where, in general, $\mathbf{f}_i(t)=(0,\ x_{\mathrm{sgn}(i)(|i|-1)}^{(|i|-2)})^T$, and $\mathrm{sgn}(i)$ is the sign of i.

The solution of Eq. (8) is (e.g., Ref. [16])

$$\eta_{\pm2}^{(1)}(t)=e^{At}\eta_{\pm2}^{(1)}(0)+e^{At}\int_0^t e^{-As}\mathbf{f}_{\pm2}(s)ds,\tag{9}$$

so, finally we have

$$\eta_{\pm2}=\epsilon\eta_{\pm2}^{(1)}+O(\epsilon^2).$$

Due to relations (7), the expansions for the oscillators with $|k|>2$ start with $O(\epsilon^2)$ terms, so,

$$x_k=\epsilon^2x_k^{(2)}+O(\epsilon^3),$$

$$p_k=\epsilon^2p_k^{(2)}+O(\epsilon^3),\quad|k|>2.\tag{10}$$

We insert the expansions (10) into the equations of motion, and, for the second-order terms, we get

$$\dot{x}_k^{(2)}=p_k^{(2)},\tag{11a}$$

$$\dot{p}_k^{(2)}=-\omega_p^2x_k^{(2)}\quad(|k|>3),\tag{11b}$$

$$\dot{x}_{\pm3}^{(2)}=p_{\pm3}^{(2)},\tag{11c}$$

$$\dot{p}_{\pm3}^{(2)}=-\omega_p^2x_{\pm3}^{(2)}+x_{\pm2}^{(1)}.\tag{11d}$$

For Eqs. (11a), (11b) we select as before the trivial solution $\eta_k^{(2)}=0,\ \forall\ |k|>3$. From Eqs. (11c), (11d) we obtain

$$\eta_{\pm 3} = \epsilon^2 \eta_{\pm 3}^{(2)} + O(\epsilon^3),$$

and we get for $\eta_{\pm 3}^{(2)}$ a solution similar to Eq. (9). In general, $\forall\ l<|k|-1$, we have

$$\eta_k^{(l)} = 0;$$

and, in a similar manner as in Eq. (9), we get

$$\eta_k^{(|k|-1)}(t) = e^{At}\eta_k^{(|k|-1)}(0) + e^{At}\int_0^t e^{-As}\mathbf{f}_k(s)ds. \quad (12)$$

So, we conclude that the expansion of the solution of Eq. (2) for the noncentral oscillators is

$$\eta_k = \epsilon^{|k|-1}\eta_k^{(|k|-1)} + O(\epsilon^{|k|}). \quad (13)$$

We perform the action-angle canonical transformation to the initial system (2) for the three central oscillators, and since H_0, being integrable, depends only on the actions, the system becomes

$$\dot{x}_k = \frac{\partial H}{\partial p_k} = p_k, \quad (14a)$$

$$\dot{p}_k = -\frac{\partial H}{\partial x_k} = -V_k'(x_k) + \epsilon(x_{k+1} - 2x_k + x_{k-1}) \quad (|k|>1), \quad (14b)$$

$$\dot{w}_i = \frac{\partial H}{\partial J_i} = \omega_i + \epsilon\frac{\partial H_1}{\partial J_i}, \quad (14c)$$

$$\dot{J}_i = -\frac{\partial H}{\partial w_i} = -\epsilon\frac{\partial H_1}{\partial w_i}, \quad (|i|\le 1), \quad (14d)$$

where $\omega_i = (\partial H_0)/(\partial J_i)$ are the frequencies of the unperturbed motion of the central oscillators. Note that Eqs. (14c, 14d), for $\epsilon=0$, have the solution

$$w_i = \omega_i t + \vartheta_i, \quad (15a)$$

$$J_i = \text{const}, \quad (15b)$$

where $\vartheta_i = w_i(0)$ are the initial angles. Due to expansion (13), it holds that

$$\eta_k(T) - \eta_k(0) = \epsilon^{|k|-1}[\eta_k^{(|k|-1)}(T) - \eta_k^{(|k|-1)}(0)] + O(\epsilon^{|k|}).$$

Similarly, by integrating Eq. (14d) with respect to time for one period, we have

$$J_i(T) - J_i(0) = \epsilon\int_0^T \frac{\partial H_1}{\partial w_i}\ dt + O(\epsilon^2),$$

where, in first-order terms with respect to ϵ, the integration is performed along the unperturbed periodic orbit. Consequently, we define the periodicity conditions as

$$\epsilon^{-|k|+1}[x_k(T) - x_k(0)] = x_k^{(|k|-1)}(T) - x_k^{(|k|-1)}(0) + O(\epsilon) = 0,$$

$$\epsilon^{-|k|+1}[p_k(T) - p_k(0)] = p_k^{(|k|-1)}(T) - p_k^{(|k|-1)}(0) + O(\epsilon) = 0 \quad (|k|>1),$$

$$w_i(T) - w_i(0) = \omega_i T + O(\epsilon) = 2\pi k_i,$$

$$\frac{1}{\epsilon}[J_i(T) - J_i(0)] = \int_0^T \frac{\partial H_1}{\partial w_i}\ dt + O(\epsilon) = 0 \quad (|i|\le 1). \quad (16)$$

The basic idea of Poincaré is the division by a suitable power of ϵ. This division is consistent, because an orbit being continued for small enough $\epsilon\neq 0$ must fulfill Eq. (16) in its original form without division, thus, since $\epsilon\neq 0$, division is possible, and the condition must be still valid in the limit of $\epsilon\to 0$ because of continuity. Thus for $\epsilon\to 0$ we get the conditions for periodic motion.

$$x_k^{(|k|-1)}(T) - x_k^{(|k|-1)}(0) = 0, \quad (17a)$$

$$p_k^{(|k|-1)}(T) - p_k^{(|k|-1)}(0) = 0 \quad (|k|>1), \quad (17b)$$

$$\omega_i T = 2\pi k_i, \quad (17c)$$

$$\int_0^T \frac{\partial H_1}{\partial w_i}\ dt = 0 \quad (|i|\le 1). \quad (17d)$$

By taking into consideration Eq. (12), Eqs. (17a, 17b) become

$$\eta_k^{(|k|-1)}(T) - \eta_k^{(|k|-1)}(0) = 0 \Leftrightarrow (e^{AT} - I)\eta_k^{(|k|-1)}(0) = -e^{AT}\int_0^T e^{-As}\mathbf{f}_k(s)ds.$$

So, in order to obtain initial conditions $\eta(0)$ for periodic motion, it must hold that $|\ e^{AT} - I| \neq 0$, or

$$e^{\pm i\omega_p T} \neq 1 \Leftrightarrow T \neq \frac{2\pi}{\omega_p}n = nT_p, \quad \forall\ n\in\mathbb{Z}, \quad (18)$$

which is the nonresonance condition with the phonons of the system. We finally get

$$\eta_k^{(|k|-1)}(0) = -(e^{AT} - I)^{-1}e^{AT}\int_0^T e^{-As}\mathbf{f}_k(s)ds.$$

Equation (17c) coincides with the resonance condition (3). On the other hand, let

$$\langle H_1\rangle = \frac{1}{T}\int_0^T H_1\ \ dt$$

be the average value of H_1 along an unperturbed T-periodic orbit and let $\phi_i = k_0\vartheta_i - k_i\vartheta_0$, $i=\pm 1$ be the resonant angles. A pair of values ϕ_i defines a periodic orbit on the three-dimensional resonant torus of the unperturbed system. Since H_1, evaluated on a T-periodic orbit of the unperturbed system, is a T-periodic function of time, its average value depends only on the particular orbit and not on the initial point. So, $\langle H_1\rangle$ must depend on ϑ_i only through $\phi_{\pm 1}$. Due to Eq. (15a), it holds that

$$\frac{1}{T}\int_0^T \frac{\partial H_1}{\partial w_i}\,dt = \frac{\partial \langle H_1\rangle}{\partial \vartheta_i}, \tag{19}$$

so, since

$$\frac{\partial \langle H_1\rangle}{\partial \vartheta_i} = \sum_j \frac{\partial \langle H_1\rangle}{\partial \phi_j}\frac{\partial \phi_j}{\partial \vartheta_i} = k_0 \frac{\partial \langle H_1\rangle}{\partial \phi_i},$$

$$\frac{\partial \langle H_1\rangle}{\partial \vartheta_0} = \sum_j \frac{\partial \langle H_1\rangle}{\partial \phi_j}\frac{\partial \phi_j}{\partial \vartheta_0} = -\sum_j k_j \frac{\partial \langle H_1\rangle}{\partial \phi_j}, \quad j=\pm 1, \tag{20}$$

Eq. (17d) yields

$$\frac{\partial \langle H_1\rangle}{\partial \phi_i} = 0, \quad i=\pm 1. \tag{21}$$

According to the implicit function theorem (e.g., Ref. [17]), for analytic continuation of the periodic orbits for $\epsilon \neq 0$, the Jacobian matrix of the periodicity conditions must be invertible for $\epsilon = 0$. This matrix decomposes in 2×2 blocks along the diagonal. So the invertibility condition for the kth noncentral oscillator is

$$\begin{vmatrix} \dfrac{\partial x_k(T)}{\partial x_k(0)} - 1 & \dfrac{\partial x_k(T)}{\partial p_k(0)} \\ \dfrac{\partial p_k(T)}{\partial x_k(0)} & \dfrac{\partial p_k(T)}{\partial p_k(0)} - 1 \end{vmatrix} \neq 0. \tag{22}$$

The matrix

$$\begin{pmatrix} \dfrac{\partial x_k(T)}{\partial x_k(0)} & \dfrac{\partial x_k(T)}{\partial p_k(0)} \\ \dfrac{\partial p_k(T)}{\partial x_k(0)} & \dfrac{\partial p_k(T)}{\partial p_k(0)} \end{pmatrix},$$

however, is the monodromy matrix of the system of linearized equations for the kth noncentral oscillator, which is e^{AT}. So, condition (22) becomes

$$|e^{AT} - I| \neq 0, \tag{23}$$

which coincides to the condition (18) of the periodic orbit to exist.

The condition for the central oscillators, after an appropriate permutation of rows and columns, transforms to the following:

$$\det\begin{vmatrix} \dfrac{\partial^2 \langle H_1\rangle}{\partial \phi_i \partial \phi_j} & \dfrac{\partial^2 \langle H_1\rangle}{\partial \phi_i \partial J_k} \\ 0 & \dfrac{\partial^2 H_0}{\partial J_l \partial J_k} \end{vmatrix} \neq 0, \quad i,j \in \{-1,1\}, \quad k,l \in \{-1,0,1\},$$

which reduces to the following nondegeneracy (anharmonicity) condition of the integrable part H_0 of the Hamiltonian,

$$\det\left|\frac{\partial^2 H_0}{\partial J_i \partial J_j}\right| \neq 0, \quad i,j \in \{-1,0,1\}, \tag{24}$$

and the condition

$$\det\left|\frac{\partial^2 \langle H_1\rangle}{\partial \phi_i \partial \phi_j}\right| \neq 0, \quad i,j \in \{-1,1\}, \tag{25}$$

where, relations (20), have been taken under consideration. Equation (25) signifies that the zeros in Eq. (21) must be simple. Thus we have proved that, under conditions (23)–(25), the uncoupled periodic motion defined by Eq. (21) is continued in an open interval $(-\epsilon_0, \epsilon_0)$ of ϵ around zero for a particular value of the total energy of the oscillators. Since the resonance condition (3) is valid on a monoparametric family of invariant tori of the uncoupled system with respect to the energy, we actually prove, for a fixed $\epsilon \in (-\epsilon_0, \epsilon_0)$, the existence of *monoparametric families* of breathers. One may consider the energy or the period of the breather as a parameter along the family. This result is similar to the one proved in Ref. [12], but it is obtained in a more direct and insightful way.

It is evident from the above analysis that if only one oscillator moves in a periodic orbit for $\epsilon = 0$, H_1 would depend only on the corresponding angle, and its average value would be a function only of the corresponding action, so the present continuation method would not be applicable.

We have to point out here that by using this method, we prove at the same time the localization of the solution, because due to Eq. (13), $\lim_{k\to\pm\infty}\eta_k \to 0$, since $\eta_k^{(k-1)}$ is periodic and [as can be seen in Eq. (12)], it is bounded in the time interval $[0,T]$.

As it has already been mentioned, our method is valid for any number of central oscillators larger than one.

III. STABILITY OF THE BREATHER SOLUTIONS

It is futile to speak about Lyapunov stability, since we study a Hamiltonian system of more than two degrees of freedom, where Arnold diffusion takes place. Instead, we investigate the linear stability of the continued periodic solutions.

For $\epsilon = 0$ the monodromy matrix of the linearized system consists of 2×2 sub-blocks. Because of the symplectic character of these sub-blocks, the eigenvalues of the central oscillators lie at unity, while the rest lie on the unit circle at two conjugate points, as mentioned before, different from 1. By setting the perturbation $\epsilon \neq 0$, the eigenvalues of the noncentral oscillators move along the unit circle, since they are of the same kind in the sense of Krein theory [18]. The eigenvalues of the central oscillators become

$$\lambda_i = e^{\pm \sigma_i T},$$

where σ_i are the *characteristic exponents*. According to Ref. [9], Sec. 79, σ_i are analytic with respect to $\sqrt{\epsilon}$, so they are expanded as

$$\sigma_i=\sqrt{\epsilon}\sigma_{i1}+o(\sqrt{\epsilon}),\quad i\in-1,0,1, \tag{26}$$

where σ_{i1}^2 are the eigenvalues of the 3×3 matrix

$$E_{ik}=-\sum_{j=-1}^{1}\frac{\partial^2\langle H_1\rangle}{\partial\vartheta_i\partial\vartheta_j}\frac{\partial^2 H_0}{\partial J_j\partial J_k},\quad i,k\in\{-1,0,1\}. \tag{27}$$

The existence of the other oscillators affects the expansion of the exponents in order higher than $\sqrt{\epsilon}$.

Due to conservation of energy, one pair of eigenvalues λ_i of the central oscillators remains equal to 1, i.e., one pair of exponents remains equal to zero in the perturbed system. If the other eigenvalues lie on the unit circle of the complex plane (i.e., the corresponding exponents are purely imaginary), the breather is linearly stable, while if they have modulus different from 1 it will be unstable. If all nonzero σ_{i1}^2 are negative and are simple eigenvalues of the above matrix **E**, complex instability cannot occur due to higher-order terms, since, in this case a quadruple of complex eigenvalues λ_i should be formed, with each pair in the neighborhood of $1\pm\sqrt{\epsilon}\sigma_{i1}$. This is, however, impossible for ϵ sufficiently small if σ_{i1}^2 is a simple eigenvalue of **E**. Since, as it can be shown (Appendix A), only one pair of characteristic exponents remains zero, the continued periodic orbits are isolated.

IV. AN EXAMPLE

A. The Morse oscillator

The Morse oscillator is defined by the potential $V_M(x)=(e^{-x}-1)^2$, and its Hamiltonian is

$$H_M=\frac{1}{2}p^2+(e^{-x}-1)^2. \tag{28}$$

The action-angle canonical transformation is defined in the domain of bounded motion, and for this system it is given by

$$w=\arccos\left(\frac{1-(1-E)e^x}{\sqrt{E}}\right),$$

$$J=\sqrt{2}(1-\sqrt{1-E}), \tag{29}$$

where E is the energy of the oscillator, i.e., the value of H_M for a specific bounded orbit. The Hamiltonian in action-angles variables becomes

$$H_M=\frac{1}{2}(2\sqrt{2}J-J^2).$$

The frequency of the oscillator is

$$\omega=\sqrt{2(1-E)}, \tag{30}$$

and the solution is

$$x(t)=\ln\left\{\frac{1-\sqrt{E}\cos(\sqrt{2(1-E)}t+\vartheta)}{1-E}\right\}. \tag{31}$$

Note that for periodic motion, it holds that $0<E<1$. The value $E=0$ corresponds to the stable equilibrium at $x=0$, while $E=1$ corresponds to the unstable equilibrium at infinity and its separatrix.

B. Multibreathers in a chain of coupled Morse oscillators

We define the chain by a countable set of coupled Morse oscillators. The Hamiltonian of the full system is

$$H=H_0+\epsilon H_1=\sum_{i=-\infty}^{\infty}\left(\frac{1}{2}\;p_i^2+(e^{-x_i}-1)^2\right)$$
$$+\frac{\epsilon}{2}\sum_{i=-\infty}^{\infty}(x_{i+1}-x_i)^2,$$

and the equations of motion of the kth oscillator are

$$\dot{x}_k=p_k,$$

$$\dot{p}_k=2(e^{-x_k}-1)e^{-x_k}+\epsilon(x_{k+1}-2x_k+x_{k-1}).$$

For $\epsilon=0$, we assume that all the oscillators lie on the stable equilibrium $(x_k,p_k)=(0,0)$, except the three central ones, which move in periodic orbits, satisfying the resonance condition $k_{-1}T_{-1}=k_0T_0=k_1T_1=T$. In order to compute the periodic orbits of the unperturbed system which will be continued for $\epsilon\neq0$, we have, first of all, to find the solutions of Eq. (21) for the specific Hamiltonian. In Appendix B, we compute the average value of H_1,

$$\langle H_1\rangle=\frac{1}{T}\int_0^T H_1\;dt, \tag{32}$$

where we remind that the integration is performed for $\epsilon=0$. Since the solution (31) is known, the use of action-angle variables is not necessary. We find

$$\langle H_1\rangle=\sum_{i=\pm1}\frac{2}{k_0k_i}\int\arctan\left(\frac{\sin\phi_i}{z_i-\cos\phi_i}\right)d\phi_i,$$

where $z_i=e^{k_ia_0+k_0a_i}$ and $\cosh a_i=E_i^{-1/2}$. The orbits that will be continued are those which satisfy

$$\frac{\partial\langle H_1\rangle}{\partial\phi_i}=0\Rightarrow\phi_i=0,\pi.$$

These solutions also satisfy $\det|\partial^2\langle H_1\rangle/\partial\phi_i\partial\phi_j|\neq0$ and the continuation is valid.

To define the stability of the breather solution, we need to calculate the matrix **E** defined in Eq. (27). Since in this case

$$\frac{\partial^2 H_0}{\partial J_l\partial J_j}=-\delta_{lj},$$

it holds that

$$E_{ij}=\frac{\partial^2\langle H_1\rangle}{\partial\vartheta_i\partial\vartheta_j},\quad\forall\quad i,j\in\{-1,0,1\},$$

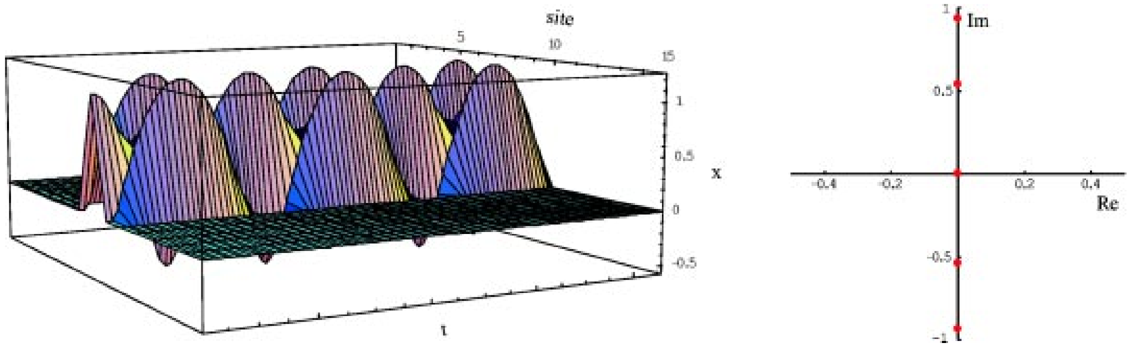

FIG. 1. Time evolution of a stable multibreather, with 1:1 resonance, for three periods and the corresponding σ_{i1} which lie on the imaginary axis.

and finally we get

$$E_{ij}=\frac{\partial^2\langle H_1\rangle}{\partial\vartheta_i\partial\vartheta_j}=\begin{cases} k_0^2\dfrac{\partial^2\langle H_1\rangle}{\partial\phi_i\partial\phi_j}, & i,j\neq 0,\\ -k_0\sum_l k_l\dfrac{\partial^2\langle H_1\rangle}{\partial\phi_i\partial\phi_l}, & i\neq 0, j=0 \quad l=\pm 1,\\ \sum_{l,m} k_l k_m\dfrac{\partial^2\langle H_1\rangle}{\partial\phi_l\partial\phi_m}, & i,j=0 \quad l,m=\pm 1. \end{cases} \tag{33}$$

This matrix is obviously symmetric with

$$E_{11}=\frac{2k_0}{k_1}\ \frac{z_1\cos\phi_1-1}{z_1^2-2z_1\cos\phi_1+1},$$

$$E_{10}=-2\ \frac{z_1\cos\phi_1-1}{z_1^2-2z_1\cos\phi_1+1},$$

$$E_{-1-1}=\frac{2k_0}{k_{-1}}\ \frac{z_{-1}\cos\phi_{-1}-1}{z_{-1}^2-2z_{-1}\cos\phi_{-1}+1},$$

$$E_{-10}=-2\ \frac{z_{-1}\cos\phi_{-1}-1}{z_{-1}^2-2z_{-1}\cos\phi_{-1}+1},$$

$$E_{00}=\frac{2k_1}{k_0}\ \frac{z_1\cos\phi_1-1}{z_1^2-2z_1\cos\phi_1+1}+\frac{2k_{-1}}{k_0}\ \frac{z_{-1}\cos\phi_{-1}-1}{z_{-1}^2-2z_{-1}\cos\phi_{-1}+1},$$

$$E_{-11}=0.$$

Next we calculate the eigenvalues of the matrix **E**, which coincide to the square of σ_{i1} in Eq. (26). Apart from one zero σ_{i1}^2, if either ϕ_1 or ϕ_{-1} equal to π, the corresponding eigenvalue is negative, which gives a pair of imaginary exponents, while, if either ϕ_1 or ϕ_{-1} is equal to 0, the corresponding eigenvalue is positive, and supplies a pair of real exponents. So, the only case of linearly stable breather solution is $\phi_1=\phi_{-1}=\pi$.

The next step is to determine the resonance between the central oscillators. We usually want to have symmetric solutions, so we choose $k_{-1}=k_1$ and $\omega_{-1}=\omega_1$, but we could obtain nonsymmetric solutions also. In the symmetric case, the nonzero eigenvalues of **E** are

$$\sigma_{i1}^2=\left\{\frac{2k_0}{(1+\beta_{k_0}^{k_1}\beta_{k_1}^{k_0})k_1},\frac{2(k_0^2+2k_1^2)}{(1+\beta_{k_0}^{k_1}\beta_{k_1}^{k_0})k_0k_1}\right\},$$

with

$$\beta_k=\exp\left[\operatorname{arccosh}\left(\sqrt{\frac{2}{2-k^2\omega^2}}\right)\right],$$

which are distinct, so, for sufficiently small ϵ, complex instability due to higher-order terms in Eq. (26) cannot occur.

For every resonance there is a family of continued periodic orbits. We choose one of these periodic orbits by fixing the energy of the oscillators through their frequencies (ω_0,ω_1). We calculate the initial conditions of the unperturbed periodic orbits that will be continued for $\epsilon\neq 0$ through Eqs. (28)–(30) with a free variable, which defines the starting point on the particular orbit. In this way we define the ϵ neighborhood of the initial conditions of the breather solution. Then we approximate the accurate initial conditions numerically. In Fig. 1 a representative breather of the 1:1 resonance for $\epsilon=0.01$ and the corresponding σ_{i1} are shown.

V. DISCUSSION

We have proven the existence of families of multibreather solutions in chains of coupled one degree of freedom Hamiltonian oscillators. We based this proof on a modification of a theorem by Poincaré [8,9]. At the same time, we calculated the ϵ neighborhood of the initial conditions for these solutions, as well as their linear stability. Finally, we applied these results to the case of a chain consisting of coupled Morse oscillators.

For the proof we considered a Hamiltonian of the form $H=H_0+\epsilon H_1$, where H_1 describes nearest-neighbor coupling and is independent of ϵ. The results, however, apply as well for every perturbation, analytic with respect to the parameter ϵ. In this case, the zero-order term of the expansion of H_1 with respect to ϵ shall be considered.

We explained why we cannot prove the existence of one-site breathers in chains of coupled one degree of freedom oscillators. But our method applies for one-site breathers also, in the case of integrable oscillators with more than one degrees of freedom. If we would use a chain of coupled n degrees of freedom oscillators, we could consider only the central oscillator on a periodic orbit of a n-dimensional resonant torus, and redefine the resonance condition (3) to be

$$\frac{\omega_1}{k_1} = \cdots = \frac{\omega_n}{k_n} = \omega.$$

Here ω_i are the frequencies of the various degrees of freedom of the specific oscillator. In this way, one could prove the existence of breatherlike solutions in multidimensional networks of oscillators of more than one degree of freedom, with period $T = 2\pi/\omega$.

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APPENDIX A: ISOLATION OF THE CONTINUED PERIODIC ORBITS

To prove that only one pair of characteristic exponents remains zero for $\epsilon \neq 0$, it is sufficient to prove that

$$\mathrm{rank}(\mathbf{E}) = 2.$$

This reduces, due to Eq. (24), to the following:

$$\mathrm{rank}(\mathbf{A}) = \mathrm{rank}\left(\frac{\partial^2 \langle H_1 \rangle}{\partial \vartheta_i \partial \vartheta_j}\right) = 2.$$

Since H_1 is single valued, $\int_T dH_1 = 0$, evaluated on a T-periodic orbit. However,

$$\int_T dH_1 = \int_0^T \frac{dH_1}{dt} dt = \int_0^T [H_1, H_0] dt = \omega_i \int_0^T \frac{\partial H_1}{\partial w_i} dt,$$

or, by Eq. (19),

$$\omega_i \frac{\partial \langle H_1 \rangle}{\partial \vartheta_i} = 0.$$

Differentiating with respect to ϑ_j,

$$\omega_i \frac{\partial^2 \langle H_1 \rangle}{\partial \vartheta_i \partial \vartheta_j} = 0. \tag{A1}$$

Since $\omega_i \neq 0$ on the resonant torus, the matrix $\mathbf{A}$ has a zero eigenvalue. We multiply the jth column of the above matrix by ω_i and replace the first column with the sum of the columns. By taking under consideration Eq. (A1), we take

$$\mathrm{rank}(A_{ij}) = \mathrm{rank}\begin{pmatrix} 0 & A_{-10} & A_{-11} \\ 0 & A_{00} & A_{01} \\ 0 & A_{10} & A_{11} \end{pmatrix}.$$

This leads, using Eq. (33), to

$$\mathrm{rank}(\mathbf{E}) = \mathrm{rank}\left(\frac{\partial^2 \langle H_1 \rangle}{\partial \phi_i \partial \phi_j}\right) = 2,$$

by condition (25). Similarly, it can be proven [19] that if $\mathbf{E}$ is an $n \times n$ matrix, rank $\mathbf{E} = n-1$.

APPENDIX B: COMPUTATION OF $\langle H_1 \rangle$ FOR THE CHAIN OF MORSE OSCILLATORS

Since $(x_k, p_k) \neq 0$ only for $k = -1$, 0, 1, H_1 (evaluated in the unperturbed system) becomes

$$H_1 = \frac{1}{2}[x_{-1}^2 + (x_{-1} - x_0)^2 + (x_0 - x_1)^2 + x_1^2].$$

The x_i^2 terms in H_1 are of no interest, because their average value is a constant quantity c_0 (i.e., independent of ϕ_i). So, instead of computing the integral in Eq. (32), we only have to evaluate the following:

$$I = \int_0^T (x_{-1}x_0 + x_0 x_1) dt. \tag{B1}$$

The Fourier expansion of the solution (31) is [20]

$$x(t) = C - 2\sum_{s=1}^{\infty} s^{-1} e^{-sa} \cos[s(\sqrt{2(1-E)}\ \ t + \vartheta)]$$

or

$$x(t) = C - \sum_{s=1}^{\infty} C_s \cos(s\omega t + s\vartheta),$$

where

$$C_s = 2s^{-1}e^{-sa}, \quad \omega = \sqrt{2(1-E)}, \quad \cosh a = E^{-1/2}.$$

The expansions for x_0 and x_1 are

$$x_0(t) = C_0 - \sum_{s=1}^{\infty} C_{0,s} \cos(s\omega_0 t + s\vartheta_0),$$

$$x_1(t) = C_1 - \sum_{r=1}^{\infty} C_{1,r} \cos(r\omega_1 t + r\vartheta_1).$$

We calculate the integral $I_1 = \int_0^T x_0 x_1\, dt$, which, by taking under consideration the resonance condition (3), becomes

$$
\begin{aligned}
I_1=\int_0^T x_0 x_1\, dt=\int_0^T C_0 C_1\, dt \\
&+\sum_{s=1}^{\infty}\sum_{r=1}^{\infty} C_{0,s}C_{1,r}\int_0^T \cos(sk_0\omega t+s\vartheta_0)\cos(rk_1\omega t \\
&+r\vartheta_1)dt - C_1\sum_{s=1}^{\infty} C_{0,s}\int_0^T \cos(sk_0\omega t+s\vartheta_0)dt \\
&-C_0\sum_{r=1}^{\infty} C_{1,r}\int_0^T \cos(rk_1\omega t+r\vartheta_1)dt. \qquad \text{(B2)}
\end{aligned}
$$

The first term of Eq. (B2) provides a constant c_1, while the last two terms are zero because they are integrals of cosines over a multiple of their period. The second term can be written as

$$
\begin{aligned}
I_2=\frac{1}{2}\sum_{s=1}^{\infty}\sum_{r=1}^{\infty} C_{0,s}C_{1,r}\Bigg[\int_0^T \cos[(sk_0+rk_1)\omega t+(s\vartheta_0 \\
+r\vartheta_1)]dt+\int_0^T \cos[(sk_0-rk_1)\omega t+(s\vartheta_0-r\vartheta_1)]dt\Bigg].
\end{aligned}
$$

The quantity inside the brackets is nonzero only if $sk_0=rk_1$ or $sk_0=-rk_1$. Note that since $r,s\in\mathbb{N}$, the previous two equations cannot be simultaneously true for fixed k_i. Let $sk_0=rk_1$. Then we set

$$
s=k_1 m,
$$

$$
r=k_0 m,
$$

so, I_2 becomes

$$
I_2=\frac{1}{2}\sum_{m=1}^{\infty} C_{0,k_1 m}C_{1,k_0 m}\int_0^T \cos[m(k_1\vartheta_0-k_0\vartheta_1)]dt.
$$

The resonant angles are $\phi_1=k_0\vartheta_1-k_1\vartheta_0$ and $\phi_{-1}=k_0\vartheta_{-1}-k_{-1}\vartheta_0$, so

$$
I_1=\int_0^T x_0 x_1\, dt=\frac{T}{2}\sum_{m=1}^{\infty} C_{0,k_1 m}C_{1,k_0 m}\cos(m\phi_1)+c_1. \qquad \text{(B3)}
$$

A similar formula is obtained for $I_{-1}=\int_0^T x_0 x_{-1} dt$ and, by using Eqs. (B1)–(B3), we get

$$
\begin{aligned}
\langle H_1\rangle = -\frac{1}{2}\Bigg(\sum_{m=1}^{\infty} C_{0,k_1 m}C_{1,k_0 m}\cos(m\phi_1) \\
+\sum_{m=1}^{\infty} C_{0,k_{-1}m}C_{-1,k_0 m}\cos(m\phi_{-1})\Bigg)+c,
\end{aligned}
$$

where $c=c_0+c_1+c_{-1}$. We recall that $C_{i,s}=2s^{-1}e^{-sa_i}$, so we have

$$
\begin{aligned}
&\sum_{m=1}^{\infty} C_{0,k_{\pm1}m}C_{\pm1,k_0 m}\cos(m\phi_{\pm1}) \\
&=\frac{4}{k_0 k_{\pm1}}\sum_{m=1}^{\infty}\frac{e^{-(k_{\pm1}a_0+k_0 a_{\pm1})m}}{m^2}\cos(m\phi_{\pm1}).
\end{aligned}
$$

We put $z_{\pm1}=e^{k_{\pm1}a_0+k_0 a_{\pm1}}$ and, by using the table of sums [20], we finally get

$$
\begin{aligned}
\langle H_1\rangle = \frac{2}{k_0 k_1}\int \arctan\left(\frac{\sin\phi_1}{z_1-\cos\phi_1}\right)d\phi_1 \\
+\frac{2}{k_0 k_{-1}}\int \arctan\left(\frac{\sin\phi_{-1}}{z_{-1}-\cos\phi_{-1}}\right)d\phi_{-1},
\end{aligned}
$$

where we have dropped the terms that are independent of ϕ_i.

[1] S. Takeno, K. Kisoda, and A.J. Sievers, Prog. Theor. Phys. Suppl. **94**, 242 (1988).
[2] S. Aubry, Physica D **103**, 201 (1997).
[3] S. Flach and C.R. Willis, Phys. Rep. **295**, 181 (1998).
[4] R.S. MacKay and S. Aubry, Nonlinearity **7**, 1623 (1994).
[5] R. Livi, M. Spicci, and R.S. MacKay, Nonlinearity **10**, 1421 (1997).
[6] J.A. Sepulchre and R.S. Mackay, Nonlinearity **10**, 679 (1997).
[7] S. Aubry, G. Kopidakis, and V. Kadelburg, Discrete Contin. Dyn. Syst., Ser. B **1**, 271 (2001).
[8] H. Poincaré, Acta Math. **13**, 1 (1890).
[9] H. Poincaré, *Les Méthodes Nouvelles de la Méchanique Céleste* (Gauthier-Villars, Paris, 1892), Vol. I [English translation: *New Methods in Celestial Mechanics*, edited by D.L. Goroff (American Institute of Physics, New York, 1993)].
[10] K. Wodnar, E. Meletlidou, and S. Ichtiaroglou, Physica D **128**, 70 (1999).
[11] V. Koukouloyannis and S. Ichtiaroglou, Chaos, Solitons Fractals **13**, 1317 (2002).
[12] T. Ahn, Nonlinearity **11**, 965 (1998).
[13] T. Ahn, Nonlinearity **13**, 29 (2000).
[14] P. Binder, D. Abraimov, A.V. Ustinov, S. Flach, and Y. Zolotaryuk, Phys. Rev. Lett. **84**, 745 (2000).
[15] G.D. Birkhoff, *Dynamical Systems* (American Mathematical Society, Providence, RI, 1927).
[16] D. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations*, 2nd ed. (Clarendon Press, Oxford, 1987).
[17] J. Dieudonné, *Foundations of Modern Analysis* (Academic Press, New York, 1969).
[18] V.A. Yakubovich and V.M. Starzhinskii, *Linear Differential Equations with Periodic Coefficients* (Wiley, New York, 1975).
[19] E. Meletlidou, Ph.D. Thesis, University of Thessaloniki, 1995 (unpublished).
[20] F. Oberhettinger, *Fourier Expansions: A Collection of Formulas* (Academic Press, New York, 1973).